Indian Statistical Institute
Second Semester Examination 2004-2005
M.Math II Year
Graph Theory and Combinatorics

Time: 4 hrs | Date:07-03-05 | Max. Marks : 100

Answer as many questions as you can. But the maximum you may score is 100.

1. (a) For integers $m, n \geq 1$, let $R(m, n)$ denote the smallest number v such that any graph on v vertices has a clique of size m or a co-clique of size n. Show that $R(m, n) = R(n, m) \leq R(m, n-1) + R(m-1, n)$. Show that for equality to hold here, at least one of the numbers on the right hand side must be odd.

   (b) Show that any 8-vertex graph with no clique of size 3 and no co-clique of size 4 must be regular of degree 3. Draw such a graph and hence prove that $R(3, 4) = 8$.

   (c) Let G be the graph with vertex set $\mathbb{F}_{17} := \mathbb{Z}/17\mathbb{Z}$, such that $x, y \in \mathbb{F}_{17}$ are adjacent in G if and only if $x - y$ is a non-zero square in the field $\mathbb{F}_{17}$. Show that G is isomorphic to its complement $\bar{G}$. Show that G has an edge-transitive automorphism group. Hence prove that G has no clique nor co-clique of size 4 and $R(4, 4) = 18$.

   [5+15+15 =35]

2. (a) Let S^1 be the unit circle in $\mathbb{R}^2$ equipped with the metric d, where $d(x, y)$ is the length of the shorter circular arc joining x & y. Show that any 3-point subspace of $(S^1,\ d)$ embeds isometrically in $\mathbb{R}$ but it has a 4-point subspace which does not so embed. Conclude that $(S^1,\ d)$ does not isometrically embed into $\mathbb{R}$.

   (b) Let (X, d) be a metric space with $\#(X) \geq 4$. Suppose every 4-point subspace of X embeds isometrically into $\mathbb{R}$. Fix two distinct points $x_0, x_1, \in X$, where w.l.g. $d(x_0, x_1) = 1$. Define $f : \{x_0, x_1\} \to \mathbb{R}$ by $f(x_0) = 0, \quad f(x_1) = 1$. Show that there is a unique function $F : X \to \mathbb{R}$ such that F extends f and F is an isometry. [5+10 = 15]

3. (a) Prove that the parameters $v,\ k,\ \lambda,\ \mu$ of any strongly regular graph satisfies $k(k - \ \lambda - 1) = \mu(v - k - 1)$. Hence prove that in any primitive (i.e. connected and co-connected) strongly regular graph we have $v \leq k^2 + 1$. There are only four feasible sets of parameters $(v,\ k,\ \lambda,\ \mu)$ for which equality holds - find these.

   (b) Let G be a (10, 3, 0, 1) strongly regular graph. Count the number of induced pentagons in G. If P is one of these induced pentagons then show that the induced subgraph $\tilde{P}$ of G on $V(G) \setminus V(P)$ is again an induced pentagon. Show that there is a bijection $f : V(P) \to V(\tilde{P})$

such that $x \in V(P)$ is joined in G to $y \in V(\tilde{P})$ if and only if $y = f(x)$. Show that f maps edges of P to non-edges of $\tilde{P}$ and vice-versa. Hence draw a picture of G. [15+20 =35]

4. (a) Show that the spectrum of any bi-partite graph is symmetric about 0.

   (b) if r is the rational rank of a reduced graph G then show that G has $\leq 2^r$ vertices.

   (c) Find all reduced graphs with exactly one negative eigenvalue. [10+10+10 =30]